

LOGIC ON THE n -CUBE

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ABSTRACT. We endow the partially ordered set of nonempty faces of the n -cube with a distinguished 0-dimensional face and three operations that naturally extend the Rota-Metropolis partial operations. While the structures thus obtained turn out to be term-equivalent to Post algebras of order 3, the inclusion order between faces coincides with the De Luca-Termini sharpening order, and yields a compact coNP-complete logic that tolerates a modicum of inconsistency and nonmonotonicity.

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1. INTRODUCTION: ORDER AND ALGEBRA ON THE FACES THE n -CUBE

For all $n \geq 5$ the only two possible order structures arising from the faces of regular (convex) polyhedra in euclidean n -space are those obtained from the n -cube and the n -simplex, [9], [15, p.190].

The lattice of all faces of the $(n-1)$ -simplex ($n = 1, 2, \dots$) can be identified with the powerset of $\{1, \dots, n\}$, i.e., with the boolean algebra \mathcal{B}_n with 2^n elements.

For an analogous treatment of the set \mathcal{F}_n of *nonempty* faces of the n -cube, in [14], Rota and Metropolis endow \mathcal{F}_n with an operation \sqcup and two partial operations \sqcap, \triangle as follows:

- (i) the smallest face $A \sqcup B$ containing the faces A and B ,
- (ii) the intersection $A \sqcap B$ of any two intersecting faces A and B ,
- (iii) the “antipodal” $\triangle(B, A)$ of A in B whenever $A \subseteq B$. The vertices of $\triangle(B, A)$ are symmetric to the vertices of A with respect to the center of B .

To give a three-valued logical interpretation of \mathcal{F}_n , Rota and Metropolis consider the set of all pairs $A = (A_0, A_1)$ of disjoint subsets of $\{1, 2, \dots, n\}$, with the understanding that A_0 (resp., A_1) is the set of coordinates where all points of the face A of the n -cube constantly have value 0 (resp., value 1). The operation \sqcup is given by $(A_0, A_1) \sqcup (B_0, B_1) = (A_0 \cup B_0, A_1 \cup B_1)$. The partial operation \sqcap is defined whenever $A_0 \cap B_1 = \emptyset = A_1 \cap B_0$, by $(A_0, A_1) \sqcap (B_0, B_1) = (A_0 \cap B_0, A_1 \cap B_1)$. The partial operation \triangle is defined whenever $A_0 \supseteq B_0$ and $A_1 \supseteq B_1$, by $\triangle((B_0, B_1), (A_0, A_1)) = (B_0 \cup (A_1 \setminus B_1), B_1 \cup (A_0 \setminus B_0))$. In [14, p.694] Rota and Metropolis write:

Each face $A = (A_0, A_1)$ of the n -cube is the result of sampling a population $S = \{1, \dots, n\}$, with a view of testing the validity of a yes-no hypothesis. Here A_1 and A_0 are the subsets of S where the hypothesis does or does not hold, respectively. A third truth-value “not-yet-known” can be assigned to each element in $S \setminus (A_0 \cup A_1)$. Two results A and B of this sampling are said to be incompatible if the two faces A and B are disjoint.

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Using this representation of \mathcal{F}_n and writing for every function $f: \{1, \dots, n\} \rightarrow \{0, 1/2, 1\}$, $\iota(f) = (A_0, A_1) = (f^{-1}(0), f^{-1}(1))$, it follows that ι is a one-one correspondence (actually, an isomorphism) between the set $\{0, 1/2, 1\}^n$ of such functions and \mathcal{F}_n . We will identify via ι the two sets \mathcal{F}_n and $\{0, 1/2, 1\}^n$. Following Rota and Metropolis, by a (*finite*) *cubic algebra* we mean a partial structure $C = (C, \sqcup, \sqcap, \Delta)$ which for some integer $n \geq 1$ is isomorphic to $(\mathcal{F}_n, \sqcup, \sqcap, \Delta)$.

To give a logical interpretation of the operations \sqcup, \sqcap and Δ , in Section 2 we modify the definition of cubic algebra by extending $\Delta(x, y)$ to the total operation $\partial(x, y) = \Delta(x \sqcup y, y)$. Using our identification $\mathcal{F}_n = \{0, 1/2, 1\}^n$, we further equip every cube with two distinguished faces $1/2$ and 0 , where $1/2$ denotes the cube itself, while 0 (i.e., the constant zero function), denotes a distinguished 0-dimensional face of the cube, called *origin*. We finally replace the partial operation \sqcap by the everywhere defined operation $x \wedge y = (0 \sqcup x) \sqcap (0 \sqcup y) \sqcap (x \sqcup y)$. By definition, a *Rota-Metropolis algebra*, (for short, *RM-algebra*) is a structure with two distinguished elements $1/2$ and 0 and three everywhere defined binary operations \sqcup, ∂, \wedge , satisfying all equations satisfied by the 1-cube $(\mathcal{F}_1, 0, 1/2, \sqcup, \partial, \wedge)$.

In Theorem 3.1 we show that all operations of RM-algebras are definable in terms of the operations of Post algebras of order 3—and vice versa. It follows that the two categories of RM-algebras and Post algebras of order 3 are equivalent. As noted in Theorem 4.4, the inclusion order between faces, when interpreted in Post algebras, coincides with the De Luca-Termini “sharpening” or “enhancing” order (see [10] and references therein).

In Section 5 we introduce a consequence relation \models_\Diamond that stands to the natural inclusion order between faces of cubes as the usual consequence relation in Post logic stands to the natural order of Post algebras. We show that the resulting logic is compact, and the problem $\alpha \models_\Diamond \beta$ is coNP-complete. In sharp contrast with Post logic, \models_\Diamond is (moderately) inconsistency tolerant and non-monotonic.

2. POST ALGEBRAS OF ORDER 3

For background in universal algebra we refer the reader to [12].

Definition 2.1. ([8, §1]) A *Kleene algebra* is a distributive lattice

$$(A, 0, 1, \neg, \vee, \wedge)$$

with smallest element 0 and largest element 1 such that $\neg\neg x = x$, $\neg(x \vee y) = \neg x \wedge \neg y$, and $x \wedge \neg x \leq y \vee \neg y$.

There are many equivalent definitions of Post algebra of order 3 (see, e.g., [6, 7, 16]). In this paper we will adopt the following:

Definition 2.2. ([11, Definition 1.1]) A *Post algebra of order 3* is an algebra

$$A = (A, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$$

such that $(A, 0, 1, \neg, \vee, \wedge)$ is a Kleene algebra, $1/2 = \neg 1/2$, and for all $x \in A$, $\neg x \wedge \nabla x = \neg x \wedge x$ and $\neg x \vee \nabla x = 1$.

As noted in [8, p.242], every Kleene algebra satisfies the equation $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, whence condition (iii) in [11, Definition 1.1] is redundant.

Post algebras of order 3 are also known as “centered 3-valued Łukasiewicz algebras”. Throughout this paper, *Post algebra* will mean “Post algebra of order 3”.

Example. Let $\mathfrak{3}$ denote the set $\{0, 1/2, 1\}$. Equipping $\mathfrak{3}$ with the operations

$$\neg x = 1 - x, \quad \nabla x = \min(1, 2x), \quad x \vee y = \max(x, y), \quad x \wedge y = \min(x, y), \quad (1)$$

we obtain the Post algebra $\mathfrak{3}_{\text{Post}} = (\mathfrak{3}, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$.

Theorem 2.3. *Adopt the above notation and terminology:*

- (i) *An algebra $Q = (Q, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$ is a Post algebra iff it satisfies all equations satisfied by $\mathfrak{3}_{\text{Post}}$ iff it belongs to the equational class $HSP(\mathfrak{3}_{\text{Post}})$ generated by $\mathfrak{3}_{\text{Post}}$.*
- (ii) *Fix a cardinal $\kappa > 0$. Then the free Post algebra on κ generators is the set $\mathfrak{3}^{\mathfrak{3}^\kappa}$ of functions from the Tychonov cube $\mathfrak{3}^\kappa$ to $\mathfrak{3}$, obtainable from the constant functions $0, 1/2, 1$ and the coordinate functions $(x_1, \dots, x_\alpha, \dots) \mapsto x_\alpha$, (for each ordinal α with $0 \leq \alpha < \kappa$) by pointwise application of the operations of $\mathfrak{3}_{\text{Post}}$.*
- (iii) *Up to isomorphism, every Post algebra is the algebra of all continuous $\mathfrak{3}$ -valued functions over some totally disconnected compact Hausdorff space, with the pointwise operations of $\mathfrak{3}_{\text{Post}}$.*

Proof. (i) and (ii) follow from Birkhoff theorem [12, 4.131], together with [6, Corollary 4, p.203]. For (iii) see [6, Theorem 5, p.198], or [11, 1.6]. \square

The following binary operations on $\mathfrak{3}$ will be frequently used in this paper (the values of x are listed in the leftmost column, those of y in the top row):

$x \sqcup y$	0	1/2	1
0	0	1/2	1/2
1/2	1/2	1/2	1/2
1	1/2	1/2	1

$\partial(x, y)$	0	1/2	1
0	0	1/2	0
1/2	1	1/2	0
1	1	1/2	1

Proposition 2.4. *With reference to (1) we have*

- (i) *The operations \neg, ∇, \vee are definable in $\mathfrak{3}$ from $0, 1/2, \wedge, \partial$, and so is the operation*

$$\Delta x = \max(0, 2x - 1). \quad (2)$$

- (ii) *The binary operation $\sqcup: \mathfrak{3}^2 \rightarrow \mathfrak{3}$ is definable from $0, 1/2, \wedge, \partial$ as follows:*

$$x \sqcup y = (\neg \nabla y \wedge \nabla y \wedge 1/2) \vee (\Delta y \wedge (1/2 \vee \Delta x)) \vee \partial(0, y). \quad (3)$$

- (iii) *The binary operation $\wedge: \mathfrak{3}^2 \rightarrow \mathfrak{3}$ is not definable from $0, 1/2, \sqcup, \partial$.*

- (iv) *The algebras $\mathfrak{3}_{\text{Post}} = (\mathfrak{3}, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$ and $\mathfrak{3}_{\text{RM}} = (\mathfrak{3}, 0, 1/2, \sqcup, \partial, \wedge)$ are term-equivalent. In detail, for all $x, y \in \mathfrak{3}$ we have:*

$$\partial(x, y) = (1/2 \wedge \nabla y \wedge \nabla \neg y) \vee (\Delta x \wedge \Delta y) \vee (\nabla x \wedge \Delta \neg y), \quad (4)$$

with \sqcup given by (3). Vice versa,

$$1 = \partial(1/2, 0), \quad \neg x = \partial(1/2, x), \quad \nabla x = \partial(x, 0), \quad x \vee y = \neg(\neg x \wedge \neg y). \quad (5)$$

Proof. (i) It is easy to verify that $\neg x = \partial(1/2, x)$, $\nabla x = \partial(x, 0)$, $x \vee y = \neg(\neg x \wedge \neg y)$ and $\Delta x = \neg \nabla \neg x$.

(ii) is proved by a tedious but straightforward verification using (i).

(iii) By way of contradiction, let us suppose that \wedge is definable.

Case 1: $x \wedge y = f(x) \sqcup g(y)$ for suitable functions $f, g: \mathfrak{3} \rightarrow \mathfrak{3}$.

Then $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ whence $f(1) \sqcup g(0) = 1/2$. On the other hand, $1 \wedge 0 = 0 \neq f(1) \sqcup g(0)$, a contradiction.

Case 2: $x \wedge y = \partial(f(x), g(y))$ for suitable functions $f, g: \mathfrak{3} \rightarrow \mathfrak{3}$.

If $g(1) \in \{0, 1\}$ then $\partial(f(1/2), g(1)) \in \{0, 1\}$ while $1/2 \wedge 1 = 1/2$, a contradiction showing that $g(1) = 1/2$. It follows that $\partial(f(1), g(1)) = 1/2 \neq 1 \wedge 1$. Thus $x \wedge y \neq \partial(f(x), g(y))$, another contradiction.

(iv) follows from a straightforward computation. \square

3. RM-ALGEBRAS

Algebras in the equational class $HSP(\mathfrak{Z}_{\text{RM}})$ generated by \mathfrak{Z}_{RM} are called *RM-algebras*.

Theorem 3.1 (Equivalent categories). *With the above stipulations we have:*

- (i) *An algebra $R = (R, 0, 1/2, \sqcup, \partial, \wedge)$ is an RM-algebra iff it satisfies all equations satisfied by \mathfrak{Z}_{RM} .*
- (ii) *There is a finite set \mathcal{E} of equations involving the constants $0, 1/2$ and the operations \sqcup, ∂, \wedge such that RM-algebras can be redefined as those algebras satisfying all equations in \mathcal{E} .*
- (iii) *For any RM-algebra $B = (B, 0, 1/2, \sqcup, \partial, \wedge)$ let*

$$B' = (B, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$$

be the algebra obtained by defining $\neg, \nabla, \vee, \wedge$, and $1 = \partial(1/2, 0)$ as in (5). Then B' is a Post algebra. Conversely, for every Post algebra $A = (A, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$ let

$$A' = (A, 0, 1/2, \sqcup, \partial, \wedge)$$

be the algebra obtained by defining the operations ∂, \wedge, \sqcup as in Proposition 2.4(iv). Then A' is an RM-algebra.

- (iv) *The two categories of Post algebras and RM-algebras are equivalent, and they are also equivalent to the category of boolean algebras.*

Proof. (i) From Birkhoff theorem [12, 4.131].

(ii) We can effectively write down \mathcal{E} starting from the defining equations of Post algebras (of order 3) as given by Definition 2.2, and translating them into equations for RM-algebras using Proposition 2.4(iv).

(iii) Follows from Proposition 2.4(iv) using, if necessary, [12, 4.140].

The first statement of (iv) follows from (iii). For the rest, see [6, Theorem 8(ii), p.202]. \square

Theorem 3.2 (Representation of RM-algebras). *Let $A = (A, 0, 1/2, \sqcup, \partial, \wedge)$ be an RM-algebra.*

- (a) *Up to isomorphism, A is the algebra of all continuous \mathfrak{Z} -valued functions over some totally disconnected compact Hausdorff space, with the pointwise operations of the RM-algebra $\mathfrak{Z}_{\text{RM}} = (\mathfrak{Z}, 0, 1/2, \sqcup, \partial, \wedge) = \mathcal{F}_1$.*
- (b) *If A is finite then for some $n = 1, 2, \dots$, A has 3^n elements, and is isomorphic to the RM-algebra \mathcal{F}_n of nonempty faces of the n -cube equipped with the distinguished constants $0, 1/2$ and operations \sqcup, ∂, \wedge as follows:*
 - (i) *0 is the origin, i.e., the constant function 0;*
 - (ii) *1/2 is the cube itself, i.e., the constant function 1/2;*
 - (iii) *$x \sqcup y$ is the smallest face containing x and y ;*
 - (iv) *$\partial(x, y) = \triangle(x \sqcup y, y)$ is the antipodal face of y in $x \sqcup y$;*
 - (v) *$x \wedge y$ is the intersection of the three faces $0 \sqcup x$, $0 \sqcup y$, $x \sqcup y$. Thus, with the notation of (ii) in the Introduction,*

$$x \wedge y = (0 \sqcup x) \cap (0 \sqcup y) \cap (x \sqcup y). \quad (6)$$

- (c) For every cardinal $\kappa > 0$ the free RM-algebra on κ generators is the set $\mathfrak{Z}^{\mathfrak{Z}^\kappa}$ of all continuous \mathfrak{Z} -valued functions over the Tychonov cube \mathfrak{Z}^κ equipped with the constant functions $0, 1/2$ and the pointwise operations \sqcup, ∂, \wedge of \mathfrak{Z}_{RM} and with the coordinate functions $(x_0, \dots, x_\alpha, \dots) \mapsto x_\alpha$, for each ordinal α with $0 \leq \alpha < \kappa$ as free generators.

Proof. (a) Combine Theorems 2.3(iii) and 3.1(iii) with [12, 4.140].

(b) (i)–(iv) are immediate. Then a tedious but straightforward computation yields (6).

(c) From Theorems 2.3(ii) and 3.1(iii). \square

As a particular case of (iv) in the above theorem, $\partial(x, 0)$ is the vertex of x farthest from the origin, where the distance of a vertex v from the origin is the number of edges in a shortest path leading from 0 to v .

4. THE NATURAL INCLUSION ORDER BETWEEN FACES

The relationships between the lattice operation \wedge and the Rota-Metropolis partial operation \sqcap are deeper than what is shown in (6). To see this, proceeding as in Theorem 3.1(iii), we first equip every RM-algebra $A = (A, 0, 1/2, \sqcup, \partial, \wedge)$ with the derived constant 1 and operations \neg, ∇, \vee as follows:

$$1 = \partial(1/2, 0), \quad \neg x = \partial(1/2, x), \quad \nabla x = \partial(x, 0), \quad x \vee y = \neg(\neg x \wedge \neg y). \quad (7)$$

Definition 4.1. We say that two elements $a, b \in A$ are *compatible* if there is $c \in A$ such that $c \sqcup a = a$ and $c \sqcup b = b$. Otherwise, a, b are *incompatible*.

Proposition 4.2. If a and b are compatible elements of the RM-algebra \mathcal{F}_n of nonempty faces of the n -cube then their infimum $a \sqcap b = a \cap b$ is given by

$$a \sqcap b = (1/2 \wedge \nabla(a \wedge \neg a) \wedge \nabla(b \wedge \neg b)) \vee \neg \nabla(\neg a \wedge \neg b). \quad (8)$$

Proof. \mathcal{F}_n is the RM-algebra of all functions $f: \{1, \dots, n\} \rightarrow \mathfrak{Z}$ with the operations \mathfrak{Z}_{RM} . One now verifies (8) for each $i = 1, \dots, n$ without difficulty. \square

Proposition 4.3. Let $A = (A, 0, 1/2, \sqcup, \partial, \wedge)$ be an RM-algebra. Let the binary relation \sqsubseteq on A be given by stipulating that, for all $a, b \in A$, $a \sqsubseteq b$ iff $a \sqcup b = b$. Recalling the notation of (7) we have:

- (i) $a \sqsubseteq b$ iff $\partial(a, 0) \sqsubseteq b$ and $\partial(a, 1) \sqsubseteq b$.
- (ii) Suppose $c \in A$ is boolean, i.e., $c = \nabla c$. Then $c \sqsubseteq b$ iff $\neg c \sqcup b = 1/2$.
- (iii) $a \sqsubseteq b$ iff $\neg \partial(a, 0) \sqcup b = 1/2$ and $\neg \partial(a, 1) \sqcup b = 1/2$.

Proof. By Theorem 3.2(a), for some totally disconnected compact Hausdorff space X , A is the RM-algebra of all continuous functions $f: X \rightarrow \mathfrak{Z}$ with the pointwise operations of \mathfrak{Z} . The pointwise verification of (i)–(ii) is now immediate. (iii) is proved by a tedious but straightforward calculation. \square

Theorem 4.4. Given elements f and g in a Post algebra A of continuous functions on a boolean space X as in Theorem 2.3(iii), we say that f is *sharper* than g , and write $f \preceq g$, iff for each $x \in X$ we either have $f(x) \leq g(x) \leq \neg g(x)$ or $f(x) \geq g(x) \geq \neg g(x)$. This is the (De Luca-Termini) sharpening order [10].

We then have:

- (i) \preceq equips A with a partial order relation.
- (ii) An element $p \in A$ is \preceq -minimal iff it is boolean.
- (iii) The partial order \sqsubseteq on the RM-algebra $\mathcal{F}_n = (\mathcal{F}_n, 0, 1/2, \sqcup, \partial, \wedge)$ given by inclusion between nonempty faces of the n -cube coincides with the partial order \preceq on the Post algebra $(\mathcal{F}_n, 0, 1/2, 1, \neg, \nabla, \vee, \wedge)$ of Theorem 3.1(iii).

Proof. A tedious but straightforward verification. \square

5. THE UNDERLYING LOGIC OF RM-ALGEBRAS

Introducing RM-logic. While by Theorem 3.1, RM-algebras are an inessential variant of Post algebras (of order 3), in this section we will introduce a consequence relation arising from the De Luca-Termini sharpening order $\preceq = \sqsubseteq$ of Theorem 4.4. The resulting logic turns out to be sharply different from Post logic.

For $\mathcal{X} = \{X_1, X_2, \dots, X_\alpha, \dots\}$ a fixed but otherwise arbitrary (possibly uncountable) nonempty set of variable symbols, the set $\text{FORM}_\mathcal{X}$ of formulas is constructed in the usual way by finitely many applications of the connectives \sqcup, ∂, \wedge starting from the variables of \mathcal{X} and the constant symbols 0 and $1/2$.

A *valuation* is a function $V: \text{FORM}_\mathcal{X} \rightarrow \mathfrak{Z}$ that assigns value $1/2$ to the symbol $1/2$, value 0 to the symbol 0, and for each binary connective $* \in \{\sqcup, \partial, \wedge\}$ satisfies the identity $V(\phi * \psi) = V(\phi) * V(\psi)$. Since V is uniquely determined by its restriction $v = V \upharpoonright \mathcal{X}$, and v ranges over all elements of the set $\mathfrak{Z}^\mathcal{X}$, then every $\phi \in \text{FORM}_\mathcal{X}$ determines the function $\hat{\phi}: \mathfrak{Z}^\mathcal{X} \rightarrow \mathfrak{Z}$ given by $\hat{\phi}(v) = V(\phi)$ for all $v \in \mathfrak{Z}^\mathcal{X}$.

In particular, for each $v \in \mathfrak{Z}^\mathcal{X}$ and variable symbol $X_\alpha \in \mathcal{X}$,

$$\widehat{X_\alpha}(v) = v_\alpha, \quad (9)$$

so that $\widehat{X_\alpha}$ is the α th coordinate function on $\mathfrak{Z}^\mathcal{X}$.

Given formulas $\phi, \psi \in \text{FORM}_\mathcal{X}$ we write $\phi \equiv_\diamond \psi$ (read: ϕ is *equivalent* to ψ) if $\hat{\phi} = \hat{\psi}$. We will tacitly identify $\hat{\phi}$ with the equivalence class ϕ / \equiv_\diamond . The set $\text{FORM}_\mathcal{X} / \equiv_\diamond$ of equivalence classes is naturally equipped with the distinguished elements 0 and $1/2$ (respectively for the constant functions 0 and $1/2$ over $\mathfrak{Z}^\mathcal{X}$), as well as with the operations \sqcup, ∂, \wedge , where $\widehat{\phi * \psi} = \hat{\phi} * \hat{\psi}$ with the pointwise operation $* \in \{\sqcup, \partial, \wedge\}$ on \mathfrak{Z} . By abuse of notation, the resulting RM-algebra $\{\hat{\phi} \mid \phi \in \text{FORM}_\mathcal{X}\}$ will be denoted $\text{FORM}_\mathcal{X} / \equiv_\diamond$.

Proposition 5.1. *For any, possibly uncountable, set $\mathcal{X} \neq \emptyset$ of variables and formula $\phi \in \text{FORM}_\mathcal{X}$, let us equip $\mathfrak{Z}^\mathcal{X}$ with the product topology of the discrete set \mathfrak{Z} . It follows that $\hat{\phi}$ is continuous. Further, $\text{FORM}_\mathcal{X} / \equiv_\diamond$ is (isomorphic to) the free RM-algebra over the free generating set $\{X / \equiv_\diamond \mid X \in \mathcal{X}\}$.*

Proof. The first statement follows by induction on the number of connectives in ϕ . The second is essentially a reformulation of Theorem 3.2(c). \square

For any $\Theta \subseteq \text{FORM}_\mathcal{X}$ and $\phi \in \text{FORM}_\mathcal{X}$ we say that Θ is *incompatible* if there is a valuation $v \in \mathfrak{Z}^\mathcal{X}$ and formulas $\theta_1, \theta_2 \in \Theta$ such that $\hat{\theta}_1(v) = 1 - \hat{\theta}_2(v)$. Otherwise, Θ is *compatible*.

A moment's reflection shows that θ_1 and θ_2 are compatible iff $\hat{\theta}_1$ and $\hat{\theta}_2$ are compatible in the sense of Definition 4.1.

Definition 5.2 (RM-logic, defined via its consequence relation). We say that ϕ is a *consequence* of Θ , and we write $\Theta \models_\diamond \phi$, according to the following stipulation:

- If Θ is incompatible then every formula ψ is a consequence of Θ .
- If Θ is compatible then ϕ is a consequence of Θ iff

$$\boxed{\forall v \in \mathfrak{Z}^\mathcal{X} \exists \theta \in \Theta \cup \{1/2\} \text{ such that } \hat{\theta}(v) \sqsubseteq \hat{\phi}(v), \text{ i.e., } \hat{\theta}(v) \sqcup \hat{\phi}(v) = \hat{\phi}(v).} \quad (10)$$

In particular, $\emptyset \models_\diamond \phi$ iff $1/2 \models_\diamond \phi$ iff $\hat{\phi}$ is the constant function $1/2$ over $\mathfrak{Z}^\mathcal{X}$. In this case we write $\models_\diamond \phi$ instead of $\emptyset \models_\diamond \phi$, and say that ϕ is a *tautology*. If $\Theta = \{\theta\}$ is a singleton then for any formula ψ we write $\theta \models_\diamond \psi$ instead of $\{\theta\} \models_\diamond \psi$.

If $\mathcal{X} \subseteq \mathcal{Y}$ then $\text{FORM}_\mathcal{X} \subseteq \text{FORM}_\mathcal{Y}$, and one might wonder whether given $\Theta \subseteq \mathcal{X}$ and $\phi \in \mathcal{X}$ we should write $\Theta \models_{\diamond, \mathcal{X}} \phi$ and $\Theta \models_{\diamond, \mathcal{Y}} \phi$ to distinguish between

$\Theta \models_{\Diamond} \phi$ in $\text{FORM}_{\mathcal{X}}$ and $\Theta \models_{\Diamond} \phi$ in $\text{FORM}_{\mathcal{Y}}$. The following result shows that no such notational precaution is necessary; its proof is an immediate consequence of the definition:

Proposition 5.3. *Suppose $\mathcal{X} \subseteq \mathcal{Y}$, $\Theta \subseteq \text{FORM}_{\mathcal{X}}$ and $\phi \in \text{FORM}_{\mathcal{X}}$. Then $\Theta \models_{\Diamond, \mathcal{X}} \phi$ iff $\Theta \models_{\Diamond, \mathcal{Y}} \phi$.*

Proposition 5.4. *For any formula ϕ the following conditions are equivalent:*

- (i) ϕ is a tautology;
- (ii) both $0 \models_{\Diamond} \phi$ and $\partial(1/2, 0) \models_{\Diamond} \phi$;
- (iii) $\alpha \sqcup \neg\alpha \models_{\Diamond} \phi$ for some formula α ;
- (iv) $\beta \models_{\Diamond} \phi$ for every formula β .

Proof. Trivial. □

Proposition 5.5. *For any two formulas $\alpha, \beta \in \text{FORM}_{\mathcal{X}}$ the following conditions are equivalent:*

- (i) both α and β are tautologies;
- (ii) $\alpha \wedge \neg\alpha \wedge \beta \wedge \neg\beta$ is a tautology.

Proof. Using Proposition 5.1, let us identify $\text{FORM}_{\mathcal{X}}/\equiv_{\Diamond}$ with the free RM-algebra over the free generating set $\mathcal{X}/\equiv_{\Diamond}$, given by Theorem 3.2(c). Trivially, for every $v \in \mathfrak{Z}^{\mathcal{X}}$, $\hat{\alpha}(v) \wedge \neg\hat{\alpha}(v) \wedge \hat{\beta}(v) \wedge \neg\hat{\beta}(v) = 1/2$ iff $\hat{\alpha}(v) = \hat{\beta}(v) = 1/2$. □

Theorem 5.6 (Compactness). *Let $\Theta \subseteq \text{FORM}_{\mathcal{X}}$ be an infinite set of formulas and $\phi \in \text{FORM}_{\mathcal{X}}$. Then the following are equivalent:*

- (i) $\Theta \models_{\Diamond} \phi$.
- (ii) $\{\theta_1, \dots, \theta_k\} \models_{\Diamond} \phi$ for some $\theta_1, \dots, \theta_k \in \Theta$.

Proof. In case Θ is incompatible, both sides are true (actually, (ii) holds with $k = 2$) and hence they are equivalent.

Now suppose Θ is compatible.

(ii) \Rightarrow (i) Immediate by Definition 5.2.

(i) \Rightarrow (ii) We reformulate (10) in Definition 5.2 as follows: For every valuation $v \in \mathfrak{Z}^{\mathcal{X}}$,

- (a) If $\hat{\phi}(v) = 1$ then there is a formula $\alpha_v \in \Theta$ with $\hat{\alpha}_v(v) = 1$.
 - (b) If $\hat{\phi}(v) = 0$ then there is a formula $\beta_v \in \Theta$ with $\hat{\beta}_v(v) = 0$.
- whence
- (c) If for every $\theta \in \Theta$ we have $\hat{\theta}(v) = 1/2$ then $\hat{\phi}(v) = 1/2$.

For each valuation v the function $\hat{\alpha}_v$ is continuous (by Proposition 5.1) and hence $\hat{\alpha}_v$ has value 1 on a clopen neighbourhood $\mathcal{N}_v \ni v$. Letting v range over $\hat{\phi}^{-1}(1)$, we see that the clopen set $\hat{\phi}^{-1}(1)$ is covered by the family of neighbourhoods \mathcal{N}_v . The compactness of $\mathfrak{Z}^{\mathcal{X}}$ yields formulas $\alpha_1, \dots, \alpha_h \in \Theta$ such that

- (a') If $\hat{\phi}(v) = 1$ then there is $i \in \{1, \dots, h\}$ such that $\hat{\alpha}_i(v) = 1$.

Similarly, there are $\beta_1, \dots, \beta_k \in \Theta$ such that

- (b') If $\hat{\phi}(v) = 0$ then there is then there is $j \in \{1, \dots, k\}$ such that $\hat{\beta}_j(v) = 0$.

The compatibility of Θ ensures that its subset $\{\alpha_1, \dots, \alpha_h, \beta_1, \dots, \beta_k\}$ (is compatible and) satisfies (ii). □

Finite sets of premises. We now consider the special case when the set \mathcal{X} of variables is finite. We write FORM_m instead of $\text{FORM}_{\{X_1, \dots, X_m\}}$ (as well as instead of $\text{FORM}_{\{X_0, \dots, X_{m-1}\}}$).

Recalling Theorems 3.1 and 3.2 and we immediately have:

Proposition 5.7 (Representation). *Every function $f: \mathfrak{3}^m \rightarrow \mathfrak{3}$ has the form $f = \hat{\psi}$ for some formula $\psi \in \text{FORM}_m$. Further, $\text{FORM}_m / \equiv_\diamond$ is isomorphic to the free RM-algebra over the free generating set $\{X_1 / \equiv_\diamond, \dots, X_m / \equiv_\diamond\}$. Thus $\text{FORM}_m / \equiv_\diamond$ is isomorphic to the RM-algebra \mathcal{F}_{3^m} of nonempty faces of the 3^m -cube, with the operations \sqcup, ∂, \wedge of Theorem 3.2(b).*

Following Rota and Metropolis (see (ii) in the Introduction), the partial binary operation $\sqcap \subseteq \mathfrak{3} \times \mathfrak{3}$ is defined by

$x \sqcap y$	0	1/2	1
0	0	0	undefined
1/2	0	1/2	1
1	undefined	1	1

The following result links the consequence relation \models_\diamond with the natural inclusion order between the faces of the 3^m -cube:

Proposition 5.8. *Suppose $\Theta \subseteq \text{FORM}_m$ is compatible and $\phi \in \text{FORM}_m$.*

- (i) $\{\theta_1, \dots, \theta_k\} \models_\diamond \phi$ iff $(\hat{\theta}_1 \sqcap \dots \sqcap \hat{\theta}_k) \sqcup \hat{\phi} = \hat{\phi}$, with the pointwise operations \sqcup and \sqcap on $\mathfrak{3}$.
- (ii) In particular, writing $\text{FORM}_m / \equiv_\diamond = \{\hat{\phi} \mid \phi \in \text{FORM}_m\} = \mathcal{F}_{3^m}$, it follows that

$$\theta \models_\diamond \phi \text{ iff } \hat{\theta} \sqsubseteq \hat{\phi} \text{ iff } \hat{\theta} \sqcup \hat{\phi} = \hat{\phi} \quad (11)$$

and hence,

$$\hat{\theta} = \hat{\phi} \text{ iff } \theta \models_\diamond \phi \text{ and } \phi \models_\diamond \theta. \quad (12)$$

Proof. The proof amounts to a tedious pointwise verification using Proposition 5.7. \square

Complexity-theoretic issues in RM-logic. Mimicking (5)-(7), the derived connectives \neg, ∇, \vee are now introduced by stipulating that for all formulas ϕ and ψ ,

$$\neg\phi, \nabla\phi, \phi \vee \psi \text{ respectively stand for } \partial(1/2, \phi), \partial(\phi, 0), \neg(\neg\phi \wedge \neg\psi). \quad (13)$$

The notations $\neg\hat{\phi}, \nabla\hat{\phi}, \hat{\phi} \vee \hat{\psi}$ are self-explanatory in the light of Theorem 3.1 and Proposition 5.7.

Proposition 5.9. *It is decidable whether $\Theta = \{\theta_1, \dots, \theta_k\} \subseteq \text{FORM}_m$ is incompatible. Further, there is a Turing machine which, having in its input a compatible set $\Theta = \{\theta_1, \dots, \theta_k\} \subseteq \text{FORM}_m$, outputs a formula $\omega \in \text{FORM}_m$ such that $\hat{\omega} = \hat{\theta}_1 \sqcap \dots \sqcap \hat{\theta}_k$.*

Proof. We only prove the second statement. It suffices to assume $k = 2$. Let ω be the formula

$$\neg\nabla(\neg\theta_1 \wedge \neg\theta_2) \vee (1/2 \wedge \nabla(\theta_1 \wedge \neg\theta_1) \wedge \nabla(\theta_2 \wedge \neg\theta_2)). \quad (14)$$

Then using (7) and (13) one verifies $\hat{\omega} = \hat{\theta}_1 \sqcap \hat{\theta}_2$. \square

The following result reduces consequence to tautology in RM-logic (notation of (13)):

Proposition 5.10. *Let $\alpha, \beta \in \text{FORM}_m$. Then $\alpha \models_{\Diamond} \beta$ iff $\models_{\Diamond} (\beta \sqcup \nabla \neg \alpha) \wedge (\beta \sqcup \neg \nabla \alpha) \wedge \neg(\beta \sqcup \nabla \neg \alpha) \wedge \neg(\beta \sqcup \neg \nabla \alpha)$.*

Thus, $\hat{\alpha} = \hat{\beta}$ iff $\models_{\Diamond} (\beta \sqcup \nabla \neg \alpha) \wedge (\beta \sqcup \neg \nabla \alpha) \wedge \neg(\beta \sqcup \nabla \neg \alpha) \wedge \neg(\beta \sqcup \neg \nabla \alpha) \wedge (\alpha \sqcup \nabla \neg \beta) \wedge (\alpha \sqcup \neg \nabla \beta) \wedge \neg(\alpha \sqcup \nabla \neg \beta) \wedge \neg(\alpha \sqcup \neg \nabla \beta)$.

Proof. By (12), together with Propositions 4.3 and 5.5. \square

From 5.10 we immediately get:

Proposition 5.11. *There is a polynomial time reduction of the consequence problem $\alpha \models_{\Diamond} \beta$ to the tautology problem in RM-logic. Also the converse reduction (trivially) exists.*

The problem $\alpha \models_{\Diamond} \beta$ is as complicated as its boolean counterpart:

Theorem 5.12 (coNP-completeness of RM-consequence). *The problem $\alpha \models_{\Diamond} \gamma$ is coNP-complete, and so is the tautology problem $\models_{\Diamond} \tau$.*

Proof. First of all, the tautology problem $\models_{\text{Post}} \beta$ in Post logic is coNP-complete: to see this, after noting that the problem is in coNP, one routinely reduces to this problem the boolean tautology problem. Second, in the light of Propositions 5.5 and 5.11 it is sufficient to deal with the tautology problem $\models_{\Diamond} \beta$. Trivially the problem is in coNP. To show coNP-hardness we will reduce to it the tautology problem in Post logic. So let $\beta = \beta(X_1, \dots, X_m)$ be an arbitrary input formula in Post logic. Let the formula β' of RM-logic be obtained from β by application of the substitutions of (5). Observe that the map $\beta \mapsto \beta'$ is computable in polynomial time. Using Proposition 5.1 from β we obtain a function $\hat{\beta}': \mathbf{3}^m \rightarrow \mathbf{3}$. Let the function $f: \mathbf{3} \rightarrow \mathbf{3}$ be defined by

$$f(x) = \partial(x, 0) \sqcup \partial(\partial(0, x), 0). \quad (15)$$

Then $f(0) = 0$, $f(1/2) = 1$, $f(1) = 1/2$, and by Theorem 3.1(iii). we can write:

$$\begin{aligned} \models_{\text{Post}} \beta &\Leftrightarrow \forall v \in \mathbf{3}^m, \hat{\beta}(v) = 1 \\ &\Leftrightarrow \forall v \in \mathbf{3}^m, f(\hat{\beta}(v)) = 1/2 \\ &\Leftrightarrow \models_{\Diamond} \partial(\beta', 0) \sqcup \partial(\partial(0, \beta'), 0). \end{aligned}$$

This yields the desired reduction. \square

6. CLOSING A CIRCLE OF IDEAS: THE SIMPLEX AND THE CUBE

From the n -simplex to boolean logic. As already mentioned in the Introduction, the lattice of all faces of the $(n-1)$ -simplex ($n = 1, 2, \dots$) is isomorphic to the boolean algebra \mathcal{B}_n with 2^n elements. To give a logical formalization of \mathcal{B}_n , one first prepares m variable symbols X_1, \dots, X_m , where m is usually much smaller than n : as a matter of fact, $m = \lceil \log_2(n+1) \rceil$ variables suffice. Let FORM_m denote the set of boolean formulas in the variables X_1, \dots, X_m . Each formula $\phi(X_1, \dots, X_m)$ determines the boolean function $\hat{\phi}: \{0, 1\}^m \rightarrow \{0, 1\}$ in the usual way. In particular, for each $i = 1, \dots, m$, and m -tuple of bits $b = (b_1, \dots, b_m)$

$$\widehat{X_i}(b) = b_i, \quad (16)$$

so that $\widehat{X_i}$ is the i th coordinate function on $\{0, 1\}^m$. Fix $n = 1, \dots, 2^m$ and suppose $\Theta \subseteq \text{FORM}_m$ is satisfied by precisely n valuations. Let $\text{Mod}(\Theta) \subseteq \{0, 1\}^m$ be the set of such satisfying evaluations. Say that two formulas α, β are Θ -equivalent, and write $\alpha \equiv_{\Theta} \beta$, iff $\Theta \models \alpha \leftrightarrow \beta$. In other words, $\hat{\alpha} \upharpoonright \text{Mod}(\Theta) = \hat{\beta} \upharpoonright \text{Mod}(\Theta)$, where, as the reader will recall, the symbol \upharpoonright denotes restriction. Let

$$\text{LIND}_{\Theta} = \{\phi / \equiv_{\Theta} \mid \phi \in \text{FORM}_m\} = \{\hat{\phi} \upharpoonright \text{Mod}(\Theta) \mid \phi \in \text{FORM}_m\} \quad (17)$$

be the *Lindenbaum algebra of Θ (in boolean logic)*, i.e., the boolean algebra consisting of all \equiv_Θ -equivalence classes of formulas equipped with the operations naturally induced by the boolean connectives. Equivalently, LIND_Θ is the boolean algebra of all boolean functions on $\text{Mod}(\Theta)$ equipped with the pointwise operations \min , \max and $1 - x$.

Proposition 6.1. $\text{LIND}_\Theta \cong \mathcal{B}_n \cong \text{powerset of } \{1, \dots, n\} \cong \text{boolean algebra of faces of the } (n-1)\text{-simplex}$. If $\tau \in \text{FORM}_m$ is a tautology then LIND_τ is isomorphic to the free boolean algebra over the free generating set $\{X_1/\equiv, \dots, X_m/\equiv\}$ of coordinate functions of $\{0, 1\}^m$. The latter in turn is isomorphic to the boolean algebra of faces of the $(2^m - 1)$ -simplex \mathcal{S}_{2^m-1} (embedded in \mathbb{R}^{2^m}).

Table 1 summarizes the relationship between boolean logic and face lattices of simplexes.

BOOLEAN LOGIC IN m -VARIABLES	$(2^m - 1)$ -SIMPLEX \mathcal{S}_{2^m-1}
the set of valuations $\{0, 1\}^{\{X_1, \dots, X_m\}}$	vertices of the $(2^m - 1)$ -simplex \mathcal{S}_{2^m-1}
ϕ/\equiv , for ϕ a formula in m variables	face of \mathcal{S}_{2^m-1}
τ/\equiv , for τ a tautology in m variables	\mathcal{S}_{2^m-1} , the largest face
X_i/\equiv , for $i = 1, \dots, m$	the face of \mathcal{S}_{2^m-1} given by the vertices in $\hat{X}_i^{-1}(1)$
$\neg\phi/\equiv$	complementary face
$(\phi \vee \psi)/\equiv, (\phi \wedge \psi)/\equiv$	union, intersection of two faces
$(\phi \wedge \neg\phi)/\equiv$	the empty face
free m -generator boolean algebra	boolean algebra of faces of \mathcal{S}_{2^m-1}
$\alpha \models \beta$	$\alpha/\equiv \subseteq \beta/\equiv$
$\theta \in \text{FORM}_m$ satisfied by n valuations	$(n-1)$ -simplex $S_\theta \subseteq \mathcal{S}_{2^m-1}$
LIND_θ	boolean algebra of faces of S_θ
valuation satisfying θ	vertex of S_θ
ψ/\equiv_θ	a face of S_θ

TABLE 1. Boolean logic on the faces of the simplex.

From the n -cube to RM-logic. As explained in the Introduction, Rota and Metropolis [14] envisaged cubic algebras as the algebras of the three-valued counterpart of boolean logic arising from the set \mathcal{F}_n of nonempty faces of the n -cube ($n = 1, 2, \dots$). To write down these faces, $m = \lceil \log_3(n+1) \rceil$ variables are sufficient.

As in the case of boolean logic, it is convenient to define Lindenbaum algebras for any nonempty (possibly uncountable) set \mathcal{X} of variables, and any compatible set $\Theta \subseteq \text{FORM}_\mathcal{X}$ of formulas. To this purpose, proceeding by analogy with the boolean case, and recalling that $0, 1 \preceq 1/2$ in the sharpening order, we let the compact set $\text{Mod}(\Theta) \subseteq \{0, 1/2, 1\}^\mathcal{X}$ be defined by

$$\text{Mod}(\Theta) = \bigcap \{ \hat{\theta}^{-1}(1/2) \mid \theta \in \Theta \cup \{1/2\} \}. \quad (18)$$

This definition is reminiscent of Definition 5.2, where it is stipulated that Θ has the same consequences as $\Theta \cup \{1/2\}$.

As in (17), the *Lindenbaum algebra (in RM-logic)* LIND_Θ is now defined as the quotient of $\text{FORM}_\mathcal{X}$ by the relation $\phi \equiv_\Theta \psi \Leftrightarrow \hat{\phi} \upharpoonright \text{Mod}(\Theta) = \hat{\psi} \upharpoonright \text{Mod}(\Theta)$, with the RM-operations naturally induced by the connectives. When $\Theta = \{\theta\}$ we write LIND_θ instead of $\text{LIND}_{\{\theta\}}$. When $\Theta = \emptyset$, $\text{Mod}(\Theta) = \text{Mod}_{1/2} = \{0, 1/2, 1\}^\mathcal{X}$.

Proposition 6.2. If $\Theta = \emptyset$ then $\text{LIND}_\Theta = \text{LIND}_{1/2} = \text{FORM}_\mathcal{X} / \equiv_\diamond$. If Θ is finite, say $\Theta = \{\theta_1, \dots, \theta_h\} \subseteq \text{FORM}_m$, $\text{Mod}(\Theta)$ is a subset of $\{0, 1/2, 1\}^m$. If $\text{Mod}(\Theta)$ has $n \geq 1$ elements, LIND_Θ is isomorphic to the RM-algebra \mathcal{F}_n of the n -cube. $\text{Mod}(\Theta)$ is empty precisely when the face $\sqcap_i \hat{\theta}_i$ is a vertex of the 3^m -cube.

Proof. From Proposition 5.1. \square

For completeness, in case $\sqcap_i \hat{\theta}_i$ is a vertex of the 3^m -cube, we stipulate that LIND_θ is the *trivial* RM-algebra with one element $0 = 1/2 = 1$, alias the 0-cube, corresponding to the trivial Post algebra.

Table 2 sums up the machinery of RM-logic over finitely many variables.

RM-LOGIC	FACES OF THE CUBE
the 3^m valuations $\{0, 1/2, 1\}^{\{X_1, \dots, X_m\}}$	the 3^m dimensions of 3^m -cube \mathcal{F}_{3^m}
ϕ/\equiv_\diamond , for ϕ a formula in m variables	a face of cube \mathcal{F}_{3^m} (among 3^{3^m} faces)
τ/\equiv_\diamond , for τ a tautology in m variables	largest face of cube \mathcal{F}_{3^m} , i.e., \mathcal{F}_{3^m} itself
ϕ/\equiv_\diamond , where $\hat{\phi}(v) \in \{0, 1\} \ \forall v \in \{0, 1/2, 1\}^m$	vertex of cube \mathcal{F}_{3^m}
X_i/\equiv_\diamond , for $i = 1, \dots, m$	i th coordinate function on $\{0, 1/2, 1\}^m$
$\text{FORM}_m/\equiv_\diamond$ free m -generator RM-algebra	RM-algebra of faces of 3^m -cube \mathcal{F}_{3^m}
$\theta \models_\diamond \psi$	θ/\equiv_\diamond is a subface of ψ/\equiv_\diamond in the 3^m -cube
$(\phi \sqcup \psi)/\equiv_\diamond$	smallest face containing ϕ/\equiv_\diamond and ψ/\equiv_\diamond
$\partial(\psi, \phi)/\equiv_\diamond$	the antipodal of ϕ/\equiv_\diamond in $(\psi \sqcup \phi)/\equiv_\diamond$
$(\phi \wedge \psi)/\equiv_\diamond$	the face $\phi/\equiv_\diamond \wedge \psi/\equiv_\diamond$
$\theta \in \text{FORM}_m$ such that $\text{Mod}(\theta)$ has n elements	n -cube $\mathcal{C}_\theta = \theta/\equiv_\diamond$ as a face of the 3^m -cube
LIND_θ	RM-algebra of faces of n -cube
ϕ/\equiv_θ	a face of the n -cube
$\theta \in \text{FORM}_m$ such that $\hat{\theta}^{-1}(1/2) = \emptyset$	θ/\equiv_θ is a vertex of the 3^m -cube

TABLE 2. The n -cube and its RM-logic.

7. FINAL REMARKS AND PROBLEMS

Intuitively, the formula $\neg\phi$ in RM-logic means “ ϕ , the other way round”, in accordance with Ramsey’s view of $\neg\phi$ as the result of writing ϕ upside down, [13]. It follows that the consequence relation \models_\diamond of RM-logic has a (limited) consistency tolerance property, which Post logic does not have:

7.1. *The pair $\{\phi, \neg\phi\}$ is compatible iff ϕ is a tautology. If $\{\phi, \neg\phi\}$ is compatible then $\{\phi, \neg\phi\} \models_\diamond \psi$ iff ψ is a tautology.*

The disjunction connective \sqcup has no dual conjunction \sqcap . For, $\neg\phi \sqcup \neg\psi \equiv_\diamond \neg(\phi \sqcup \psi)$. The connective \wedge has the following consistency tolerance and nonmonotonicity properties, which disappear when \wedge is thought of as conjunction in Post logic:

7.2. *For every formula ϕ , the pair $\{\phi, \phi \wedge \neg\phi\}$ is incompatible iff $\hat{\phi}(v) = 1$ for some valuation v . In general, the set of consequences of $\alpha \wedge \beta$ is not larger than the set of consequences of α .*

Among the derived connectives of RM-logic, the “possibility” connective ∇ transforms ϕ into the “remotest possibility $\nabla\phi$ (from the origin)”, where $\nabla\phi \equiv_\diamond \nabla\nabla\phi \equiv_\diamond \partial(\phi, 0)$. We also have the “dual (nearest) possibility” $\Delta\phi$, defined as $\partial(\phi, \partial(1/2, 0))$, and satisfying $\Delta\phi \equiv_\diamond \Delta\Delta\phi \equiv_\diamond \neg\nabla\neg\phi$.

Concerning implication in RM-logic, the equivalences

$$\hat{\alpha} \subseteq \hat{\beta} \Leftrightarrow \alpha \models_\diamond \beta \Leftrightarrow \models_\diamond (\beta \sqcup \nabla\neg\alpha) \wedge (\beta \sqcup \neg\nabla\alpha) \wedge \neg(\beta \sqcup \nabla\neg\alpha) \wedge \neg(\beta \sqcup \neg\nabla\alpha)$$

of Propositions 5.8(ii) and 5.10 naturally introduce a connective \rightsquigarrow of the form

$$\alpha \rightsquigarrow \beta = (\beta \sqcup \nabla\neg\alpha) \wedge (\beta \sqcup \neg\nabla\alpha) \wedge \neg(\beta \sqcup \nabla\neg\alpha) \wedge \neg(\beta \sqcup \neg\nabla\alpha).$$

7.3. *The \rightsquigarrow connective satisfies:*

$$\text{If } \models_\diamond \alpha \text{ and } \models_\diamond \alpha \rightsquigarrow \beta \text{ then } \models_\diamond \beta. \quad (19)$$

7.4. PROBLEMS.

- (1) Analyze the negation connective \neg in RM-logic, as well the completeness and consistency properties of RM-logic in the general framework of [1, 2, 3].
- (2) Develop the proof theory of \models_{\diamond} (along the lines of [4, §5]).
- (3) Construct first-order RM-logic. Does first-order RM-logic have a nondeterministic semantics as in [5]?

REFERENCES

- [1] O.Arieli, A. Avron, A. Zamansky, Maximal and Premaximal Paraconsistency in the Framework of Three-Valued Semantics. *Studia Logica*, 97 (2011) 31-60.
- [2] A. Avron, Negation: Two points of view. In: *What is Negation? Applied Logic Series*, Vol. 13, D. Gabbay and H. Wansing, (eds.), Kluwer Academic Publishers, 1999, pp. 3-22.
- [3] A. Avron, On Negation, Completeness and Consistency. In: *Handbook of Philosophical Logic*, Vol. 9, D. Gabbay and F. Guenther, (eds.), Kluwer Academic Publishers, 2002, pp.287-319.
- [4] A. Avron, Classical Gentzen-type Methods in Propositional Many-Valued Logics. In: *Beyond Two: Theory and Applications of Multiple-Valued Logic, Studies in Fuzziness and Soft Computing*, Vol. 114, M. Fitting and E. Orłowska, (eds.), Physica Verlag, 2003, pp. 117-155.
- [5] A. Avron, A.Zamansky, Many-Valued Non-deterministic Semantics for First-Order Logics of Formal (In)consistency. In: *Algebraic and Proof-theoretic Aspects of Non-classical Logics*, Series: *Lecture Notes in Computer Science*, Vol. 4460 Subseries: *Lecture Notes in Artificial Intelligence*, S.Aguzzoli, A.Ciabattini, B.Gerla, C.Manara, V.Marra, (eds.), 2007, VIII, 309 pp.1-24.
- [6] R.Balbes, P.Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.
- [7] V.Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu, *Łukasiewicz-Moisil Algebras*, North-Holland, Amsterdam 1991.
- [8] R.Cignoli, Coproducts in the categories of Kleene and three-valued Łukasiewicz algebras, *Studia Logica*, 38 (1979) 237-245.
- [9] H.S.M. Coxeter, *Regular Polytopes*, London, Methuen, 1948. New York, Pitman, 1949. Dover, 1973.
- [10] A. de Luca, A Survey of Some Quantitative Approaches to the Notion of Information, In: *Formal and Natural Computing, Lecture Notes in Computer Science*, Springer-Verlag, Berlin Heidelberg, vol. 2300, W.Brauer et al. (eds.), (2002) 77-95.
- [11] M.Dickmann, A.Petrovich, The three-valued logic of quadratic form theory over real rings, In: *Andrzej Mostowski and Foundational Studies*, A. Mostowski, A. Ehrenfeucht, V. W. Marek, M. Srebrny, (eds.), IOS Press, 2008, pp.49-67.
- [12] R.N.McKenzie, G.McNulty, W.T.Taylor, *Algebras, Lattices, Varieties*, Vol.I, Wadsworth & Brooks/Cole, Belmont, CA, 1987.
- [13] F. P.Ramsey, Facts and Propositions, *Proceedings of the Aristotelian Society*, Suppl. Vol. 7 (1927) 153-170. Reprinted in F. P.Ramsey, *Philosophical Papers*, edited by D. H.Mellor, Cambridge: Cambridge University Press, 1990, pp. 34-51.
- [14] G-C.Rota, N.Metropolis, Combinatorial structure of the faces of the n -cube, *SIAM J. Applied Math.*, 35 (1978) 689-694.
- [15] D.M.Y.Sommerville, *An introduction to the geometry of N dimensions*, Methuen & co. Ltd., London, 1929.
- [16] A. Torrens, Cyclic elements in MV-Algebras and Post algebras, *Math. Log. Quart.*, 40 (1994) 431-444.

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